# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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CRITICAL COMPRESSIVE STRESS FOR CURVED SHEET SUPPORTED

ALONG ALL EDGES AND ELASTICALLY RESTRAINED AGAINST

ROTATION ALONG THE UNLOADED EDGES

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# NACA

## WASHINGTON

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## RESTRICTED BULLETIN

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## SUMMARY

A formula is given for the critical compressive stress
for slightly curved sheet with equal elastic restraints
against rotation along the unloaded edges. The theory of
small deflections is used and the formula reduces to that
given by Timoshenko for the case of simply-supported edges.
For larger curvatures, a modification of Redshaw's formula
to include the effect of edge restraint is suggested.

## INTRODUCTION

Because the skin between stiffeners on the surface of
airplanes is curved to the contour of the wing or fuselage,
it is important to investigate the extent to which this cur-
vature influences the critical stress.

The derivation and use of formulas for the critical
compressive stress for flat rectangular plates with re-
straints against rotation along the unloaded edges were
discussed in reference 1. In the present paper, the theory
of small deflections is used to derive a formulas for the
critical compressive stress of a slightly curved sheet with
restraints against rotation along the straight, unloaded
edges. On the basis of this derivation, a modification of
Redshaw's formula (reference 2) is suggested for cases in
which the curvature is larger.

## THEORY

Figure 1 shows the coordinate system and the sheet
dimensions. With the assumption that the sheet is curved

to a cylindrical surface, the differential equation of the deflection surface of the sheet will be that given by Donnell as equation (10) in reference 3, which is

$$D\nabla^8 w + \frac{Et}{r^2} \frac{\partial^4 w}{\partial x^4} = -\sigma_x t \nabla^4 \left( \frac{\partial^2 w}{\partial x^2} \right) \tag{1}$$

where

D     flexural stiffness of sheet per unit length $\left[ \dfrac{Et^3}{12(1-\mu^2)} \right]$

E     modulus of elasticity

$\mu$     Poisson's ratio

t     thickness of sheet

r     radius of curvature

$\sigma_x$     applied compressive stress in the x-direction

w     displacement of sheet out of its original plane

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial s^2}$, $\nabla^4 = \left( \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial s^2} \right)\left( \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial s^2} \right)$, etc.

x     longitudinal coordinate

s     circumferential coordinate

The width of the sheet is $b = r\theta$, where $\theta$ is the angle subtended by the sheet at the center of curvature. The assumption is made that the critical stress $\sigma_{cr}$ may be expressed in a manner similar to the critical stress of flat sheet, as

$$\sigma_{cr} = k_r \frac{\pi^2 E t^2}{12(1-\mu^2)b^2} \tag{2}$$

where $k_r$ is a constant which depends upon the radius of curvature and the edge restraints. Substitution of $\sigma_{cr}$,

as given by equation (2), for $\sigma_x$ in equation (1) and division by $D$ gives

$$\nabla^8 w + \frac{12(1 - \mu^2)}{r^2 t^2} \frac{\partial^4 w}{\partial x^4} = - \frac{\pi^2 k_r}{b^2} \nabla^4\left(\frac{\partial^2 w}{\partial x^2}\right) \qquad (3.)$$

On the assumption that the sheet is infinitely long in the x-direction, the solution may be taken in the form

$$w = f(s) \cos \frac{\pi x}{\lambda} \qquad (4)$$

where

$f(s)$      function of $s$ alone

$\lambda$      half-wave length of the buckle pattern

Substitution of equation (4) in equation (3) gives as the differential equation for $f(s)$

$$\frac{d^8 f}{ds^8} - 4\left(\frac{\pi}{\lambda}\right)^2 \frac{d^6 f}{ds^6} + 6\left(\frac{\pi}{\lambda}\right)^4 \frac{d^4 f}{ds^4} - 4\left(\frac{\pi}{\lambda}\right)^6 \frac{d^2 f}{ds^2} + \left(\frac{\pi}{\lambda}\right)^8 f$$

$$+ \frac{12(1 - \mu^2)}{r^2 t^2}\left(\frac{\pi}{\lambda}\right)^4 f - \frac{\pi^2 k_r}{b^2}\left[\left(\frac{\pi}{\lambda}\right)^2 \frac{d^4 f}{ds^4} - 2\left(\frac{\pi}{\lambda}\right)^4 \frac{d^2 f}{ds^2} + \left(\frac{\pi}{\lambda}\right)^6 f\right] = 0$$

The solution may be written

$$f(s) = c_1 e^{\alpha_r \frac{s}{b}} + c_2 e^{-\alpha_r \frac{s}{b}} + c_3 e^{i\beta_r \frac{s}{b}} + c_4 e^{-i\beta_r \frac{s}{b}}$$

$$+ c_5 e^{\alpha_r' \frac{s}{b}} + c_6 e^{-\alpha_r' \frac{s}{b}} + c_7 e^{i\beta_r' \frac{s}{b}} + c_8 e^{-i\beta_r' \frac{s}{b}} \qquad (5)$$

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where

$$\alpha_r = \pi \sqrt{\frac{b}{\lambda}} \sqrt{\frac{b}{\lambda} + \sqrt{\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}}$$

$$\beta_r = \pi \sqrt{\frac{b}{\lambda}} \sqrt{-\frac{b}{\lambda} + \sqrt{\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}}$$

$$\alpha_r' = \pi \sqrt{\frac{b}{\lambda}} \sqrt{\frac{b}{\lambda} + \sqrt{\frac{k_r}{2} - \sqrt{\frac{k_r^2}{4} - \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}}$$

$$\beta_r' = \pi \sqrt{\frac{b}{\lambda}} \sqrt{-\frac{b}{\lambda} + \sqrt{\frac{k_r}{2} - \sqrt{\frac{k_r^2}{4} - \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}}$$

$$(6)$$

and the c's are arbitrary constants.

As for a flat plate, there are four boundary conditions imposed on w (see reference 1). If the edges of the plate are at s = ±b/2 these conditions are

$$(w)_{s=-\frac{b}{2}} = 0$$

$$(w)_{s=\frac{b}{2}} = 0$$

$$D\left(\frac{\partial^2 w}{\partial s^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)_{s=-\frac{b}{2}} = 4S_0 \left(\frac{\partial w}{\partial s}\right)_{s=-\frac{b}{2}}$$

$$D\left(\frac{\partial^2 w}{\partial s^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)_{s=\frac{b}{2}} = -4S_0 \left(\frac{\partial w}{\partial s}\right)_{s=\frac{b}{2}}$$

where $S_0$ is the stiffness per unit length of the elastic
restraining medium or structural element at the edges of
the sheet; that is, $S_0$ is the ratio of moment per unit
length at any point along the length of the medium or struc-
ural element to the rotation in quarter-radians at that
point when the moment is distributed sinusoidally. The
requirement of a sinusoidally distributed moment is satis-
fied when an infinitely long plate buckles under longitu-
dinal compression.

Of the eight solutions summed in equation (5), only
four are required to satisfy the foregoing conditions.
The question of which solutions to use is decided by the
requirement that the deflection surface reduce to that
for a flat sheet when the radius of curvature is made to
approach infinity. Putting $r = \infty$ in equations (6) gives

$$\left.\begin{array}{l} \alpha_{\infty} = \pi \sqrt{\dfrac{b}{\lambda}} \sqrt{\dfrac{b}{\lambda} + \sqrt{k_{\infty}}} \\[12pt] \beta_{\infty} = \pi \sqrt{\dfrac{b}{\lambda}} \sqrt{-\dfrac{b}{\lambda} + \sqrt{k_{\infty}}} \\[12pt] \alpha_{\infty}' = \pi \dfrac{b}{\lambda} \\[12pt] \beta_{\infty}' = i\pi \dfrac{b}{\lambda} \end{array}\right\} \qquad (7)$$

where $k_{\infty}$ is the value of $k_r$ when $r = \infty$. The quan-
tities $\alpha_{\infty}$ and $\beta_{\infty}$ are seen to be identical with the
$\alpha$ and $\beta$ of reference 1; whereas $\alpha_{\infty}'$ and $\beta_{\infty}'$ have
no counterpart for flat plates. The deflection surface
for the curved plate may thus be written

$$w = \left( c_1 e^{\alpha_r \frac{s}{b}} + c_2 e^{-\alpha_r \frac{s}{b}} + c_3 e^{i\beta_r \frac{s}{b}} + c_4 e^{-i\beta_r \frac{s}{b}} \right) \cos \frac{\pi x}{\lambda}$$

In addition to satisfying boundary conditions involving
$w$, the effect of curvature requires that conditions involv-

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ing the displacements  u  and  v  in the longitudinal
and circumferential directions also be satisfied.  Dis-
regarding the four solutions in  $\alpha_r'$  and  $\beta_r'$  may be
interpreted physically as neglecting this effect of cur-
vature.  This neglect is permissible for plates of small
curvature.  (See reference 3.)

The theory of reference 1 shows that, for a flat
plate with an elastic restraint of stiffness  $S_0$  along
each unloaded edge, the critical buckling stress is to be
found from the expression

$$\frac{\alpha_\infty^2 + \beta_\infty^2}{\alpha_\infty \tanh \frac{\alpha_\infty}{2} + \beta_\infty \tan \frac{\beta_\infty}{2}} = - \epsilon \tag{8}$$

where  $\epsilon$  is the restraint coefficient, given as

$$\epsilon = \frac{4 S_0 b}{D}$$

The deflection surface and boundary conditions used here
are formally the same as for flat sheet; therefore, for
a curved sheet with the same restraining members

$$\frac{\alpha_r^2 + \beta_r^2}{\alpha_r \tanh \frac{\alpha_r}{2} + \beta_r \tan \frac{\beta_r}{2}} = - \epsilon \tag{9}$$

It may be seen from a comparison of the definitions of
$\alpha_\infty$, $\beta_\infty$  and  $\alpha_r$, $\beta_r$  as given in equations (7) and (6)
that, if equation (8) is satisfied when  $k_\infty$  has some
particular value, equation (9) will be satisfied when the
expression

$$\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1 - \mu^2)}{\pi^4} \left(\frac{b^2}{rt}\right)^2}$$

has this same value. A relation between the initial buck-
ling stresses for curved and for flat sheet can therefore
be obtained by setting

$$\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1 - \mu^2)}{\pi^4} \left(\frac{b^2}{rt}\right)^2} = k_\infty$$

or

$$k_r = k_\infty + \frac{12(1 - \mu^2)}{\pi^4 k_\infty} \left(\frac{b^2}{rt}\right)^2$$

By substitution of this value of $k_r$ in equation
(2) the critical stress is found to be

$$\sigma_{cr} = \left[k_\infty + \frac{12(1 - \mu^2)}{\pi^4 k_\infty} \left(\frac{b^2}{rt}\right)^2\right] \frac{\pi^2 E t^2}{12(1 - \mu^2)b^2}$$

$$= \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2} + \frac{\theta^2 E}{\pi^2 k_\infty} \tag{10}$$

where the first term is the critical compressive stress of
the sheet when flat and $k_\infty$ is the coefficient $k$ for the
flat sheet, as determined by the methods of reference 1.
For the case of simple support along the edges, $k_\infty = 4$
and equation (10) reduces to a formula given by Timoshenko
(reference 4, p. 470, equation (276)).

Redshaw, has proposed the following approximate formu-
la, derived by an energy method without limitations as to
curvature, for simply supported edges (reference 2, equa-
tion (31)):

$$\sigma_{cr} = \frac{4\pi^2 E t^2}{12(1 - \mu^2)b^2} \cdot \frac{1 + \sqrt{1 + \frac{12(1 - \mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}{2} \tag{11}$$

For curvatures so small that the radical may be expanded in series, the Redshaw formula reduces to

$$\sigma_{cr} = \frac{4\pi^2 E t^2}{12(1 - \mu^2)b^2} + \frac{\theta^2 E}{\pi^2} \tag{12}$$

which is of the same form as equation (10) but does not include the coefficient $k_\infty$ in the term that takes account of the curvature.

The formal relation between equations (11) and (12) suggests that an equation similar to equation (11) exists that will yield equation (10) when the curvature is small. Such an equation is

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2} \cdot \frac{1 + \sqrt{1 + \frac{48(1 - \mu^2)}{\pi^4}\left(\frac{b^2}{k_\infty rt}\right)^2}}{2} \tag{13}$$

Equation (13), although not actually derived, appears to be a reasonable generalization of the formula of Redshaw and yields the following results in special cases:

Flat plate; $\frac{b^2}{rt} = 0$; $\qquad \sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2}$

Slight curvature; $\frac{b^2}{rt}$ small; $\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2} + \frac{\theta^2 E}{\pi^2 k_\infty}$

Large curvature; $\frac{b^2}{rt}$ large; $\sigma_{cr} = \frac{E\frac{t}{r}}{\sqrt{12(1 - \mu^2)}} = 0.303 E\frac{t}{r}$ (for $\mu = 0.3$)

The value of $\sigma_{cr}$ for large curvature is half the classical value for a complete cylinder and is in better agreement with cylinder tests than the classical value. Equation (13) might, therefore, be expected to hold reasonably well for all curvatures and for any degree of restraint at the edges of the sheet. The tests that have been made (reference 5) indicate, however, that the effect of curvature cannot always be relied upon to follow consistently the gradual increase in critical stress with increase in curvature represented by equation (13).

## CONCLUSION

The critical compressive stress for an infinitely long slightly curved sheet with equal elastic restraints against rotation along the unloaded edges, as obtained from the differential-equation solution, is given by the equation

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2} + \frac{\theta^2 E}{\pi^2 k_\infty}$$

where the first term gives the critical compressive stress for the sheet when flat and

$k_\infty$    coefficient $k$ determined for flat sheet (by method of reference 2)

$\theta$    angle subtended by sheet at center of curvature

$E$    modulus of elasticity

$t$    thickness of sheet

$\mu$    Poisson's ratio

$b$    width of sheet

For larger curvatures, the formula

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2)b^2} \cdot \frac{1 + \sqrt{1 + \frac{48(1 - \mu^2)}{\pi^4}\left(\frac{b^2}{k_\infty r t}\right)^2}}{2}$$

is suggested, where $r$ is the radius of curvature.

The tests that have been made indicate, however, that the effect of curvature cannot always be relied upon to follow consistently the gradual increase in critical stress with increase in curvature represented by these equations.

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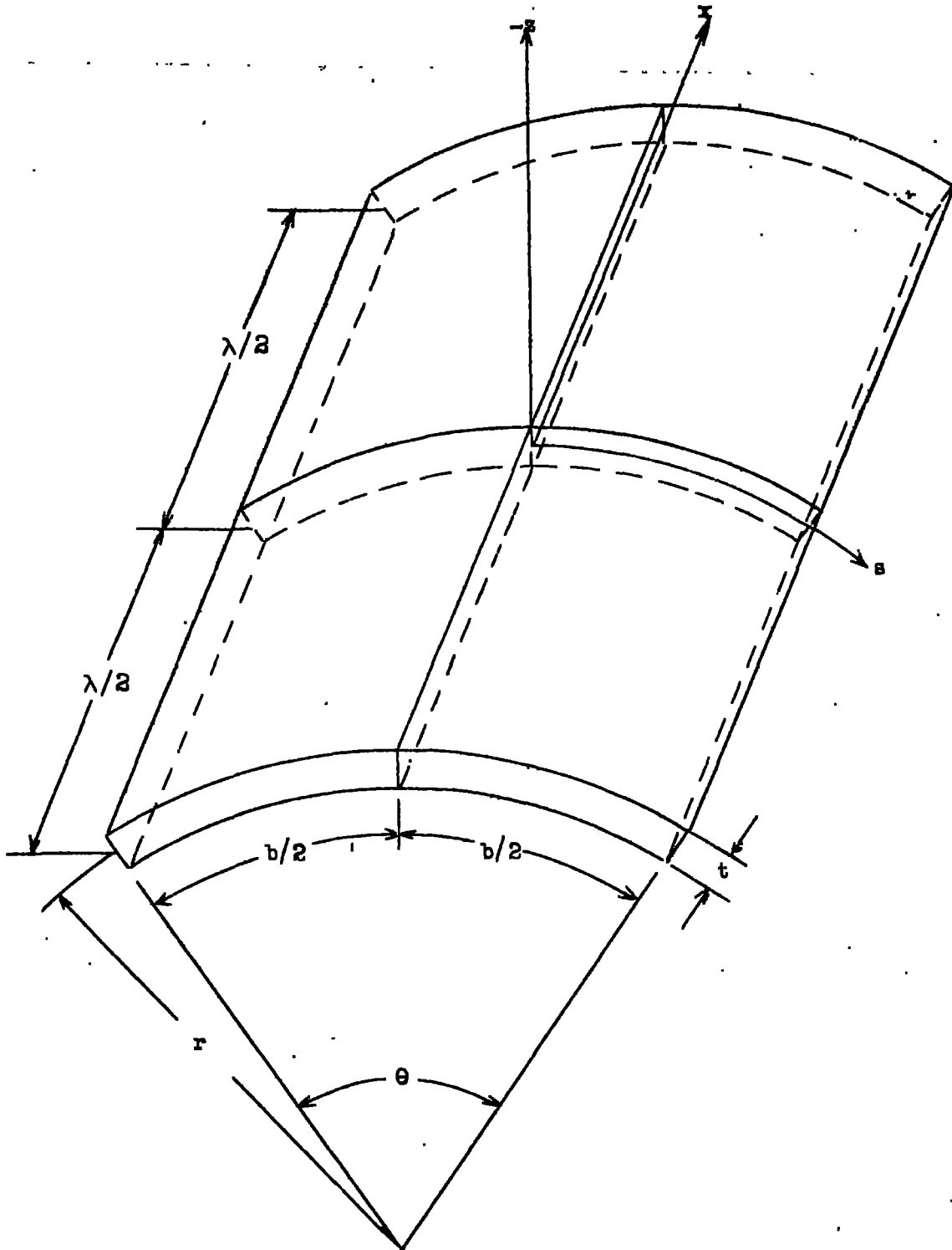
Figure 1.- Coordinate system for curved sheet.